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NON EXISTENCE OF TWISTED WAVE EQUATIONS

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ABSTRACT. Let X be a complex manifold, V an involutive submanifold of its cotangent bundle, and Σ a bicharacteristic leaf of V. A ring of twisted differential operators $\mathcal{A}$ on X has a characteristic class in $H^1(X; d\mathcal{O}_X^\times)$. To such a class we associate a class in $H^2(\Sigma; \mathbb{C}_\Sigma^\times)$ whose vanishing is necessary for the existence of an $\mathcal{A}$-module globally simple along V. As an application, we show that there are no generalized massless field equations with non trivial twist on Grassmann manifolds.

1. Statement of the problem

The grassmannian $\mathbb{G}$ of 2-dimensional planes in a 4-dimensional complex vector space $\mathbb{T}$ is a homogeneous space G/H, where $G = SL(4;\mathbb{C})$ and H is the stabilizer of a point. Let $M = (\mathbb{R}^4, \|\cdot\|)$ be the Minkowski space, where $\|(x_0, x_1, x_2, x_3)\| = x_0^2 - x_1^2 - x_2^2 - x_3^2$. According to Penrose, consider the embedding

$$\iota\colon M \to \mathbb{G}$$
$$(x_0, x_1, x_2, x_3) \mapsto \langle (x_0 + x_3, x_1 + ix_2, 1, 0), (x_1 - ix_2, x_0 - x_3, 0, 1)\rangle_{\mathbb{C}}$$

where $\langle v, w\rangle_{\mathbb{C}}$ denotes the vector space spanned by v and w. The closure of $\iota(M)$ is an orbit of the totally real form $SU(2,2) \subset SL(4;\mathbb{C})$, whose action corresponds to that of the Poincaré group on M. Thus $\mathbb{G}$ is a conformal compactification of the complexified Minkowski space. In particular, the usual wave equation, as well as the other massless field equations, extend as differential operators acting between homogeneous bundles on $\mathbb{G}$. Let us denote by $\mathcal{M}_{(m)}$ the quasi-equivariant $\mathcal{D}_{\mathbb{G}}$-module corresponding to the massless field equation of helicity $m \in \mathbb{Z}/2$. As we will recall, the modules $\mathcal{M}_{(m)}$ are "globally simple" along the characteristic variety V of the wave equation.

Denote by $\mathfrak{g} = \mathfrak{sl}(4;\mathbb{C})$ the Lie algebra of $G = SL(4;\mathbb{C})$. By the Beilinson-Bernstein-Kashiwara correspondence, quasi-equivariant $\mathcal{D}_{\mathbb{G}}$-module are associated with $(\mathfrak{g}, H)$-modules with trivial twist. In order to deal with arbitrary twists, one has to consider quasi-equivariant modules over rings of twisted differential operators on $\mathbb{G}$.

It thus arises a natural question: given a ring $\mathcal{A}$ of twisted differential operators on $\mathbb{G}$, are there $\mathcal{A}$-modules globally simple along V?

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2. Twisted sheaves and differential operators

Let us briefly review the notions of twisted sheaves and twisted differential operators from [7, 1] (see also [2] for an exposition).

Let X be a complex manifold, $\mathbb{C}_X$ the constant sheaf with stalk $\mathbb{C}$ on X, $\mathcal{O}_X$ the structure sheaf on X, and $\mathcal{D}_X$ the ring of finite order differential operators on X.

- A ring of twisted differential operators (a t.d.o. ring for short) is an $\mathcal{O}_X$-ring locally isomorphic to the ring $\mathcal{D}_X$. They are classified by $H^1(X; d\mathcal{O}_X)$, up to isomorphisms.

A basic example of t.d.o. ring is the ring

$$\mathcal{D}_{\mathcal{L}} = \mathcal{L} \otimes_{\mathcal{O}} \mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{L}^{-1}$$

of differential operators acting on a line bundle $\mathcal{L}$. For $\lambda \in \mathbb{C}$ one can also consider the t.d.o ring $\mathcal{D}_{\mathcal{L}^\lambda}$ defined as follows. If s is a nowhere vanishing local section of $\mathcal{L}$, sections of $\mathcal{D}_{\mathcal{L}^\lambda}$ are written as $s^\lambda \otimes P \otimes s^{-\lambda}$, for $P \in \mathcal{D}_X$. If t is another nowhere vanishing local section of $\mathcal{L}$, then $s^\lambda \otimes P \otimes s^{-\lambda} = t^\lambda \otimes Q \otimes t^{-\lambda}$ in $\mathcal{D}_{\mathcal{L}^\lambda}$ if and only if $Q = (s/t)^\lambda \cdot P \cdot (s/t)^{-\lambda}$ in $\mathcal{D}_X$. This is independent from the choice of a branch for the ramified function $(s/t)^\lambda$. It is also possible to give a meaning to $\mathcal{L}^\lambda$ as a twisted sheaf, as follows.

Denote by $\mathsf{Mod}(\mathbb{C}_X)$ the category of sheaves of $\mathbb{C}$-vector spaces on X, and by $\mathfrak{Mod}(\mathbb{C}_X)$ the corresponding $\mathbb{C}$-stack, $U \mapsto \mathsf{Mod}(\mathbb{C}_U)$.

- A stack of twisted sheaves is a $\mathbb{C}$-stack $\mathfrak{S}$ on X locally $\mathbb{C}$-equivalent to the stack of sheaves $\mathfrak{Mod}(\mathbb{C}_X)$. They are classified by $H^2(X; \mathbb{C}_X^\times)$, up to $\mathbb{C}$-equivalences. Twisted sheaves are objects of $\mathfrak{S}(X)$.

For an open covering $X = \bigcup_i U_i$, let $c_{ijk} \in \mathbb{C}_X^\times(U_{ijk})$ be a Cech cocycle for the characteristic class of $\mathfrak{S}$ in $H^2(X; \mathbb{C}_X^\times)$. Twisted sheaves in $\mathfrak{S}(X)$ are described by a family of sheaves F_i on U_i, and a family of isomorphisms $\theta_{ij}\colon F_j|_{U_{ij}} \to F_i|_{U_{ij}}$, satisfying $\theta_{ij} \circ \theta_{jk} = c_{ijk}\theta_{ik}$ on U_{ijk}.

For $\mathcal{B}$ a sheaf of $\mathbb{C}$-algebras, let $\mathsf{Mod}(\mathcal{B}; \mathfrak{S})$ be the category of $\mathcal{B}$-modules in $\mathfrak{S}$.

- Twisted line bundles are object of $\mathsf{Mod}(\mathcal{O}_X; \mathfrak{S})$ locally isomorphic to $\mathcal{O}_X$.

The twisted sheaf $\mathcal{L}^\lambda$ is an example of a twisted line bundle. Its twist is described as follows. Let s_i be non vanishing sections of $\mathcal{L}$ on U_i. Then $\mathcal{L}^\lambda$ belongs to a stack of twisted sheaves whose cocycle c_{ijk} describes the difference of determinations between the ramified functions $(s_i/s_j)^\lambda (s_j/s_k)^\lambda$ and $(s_i/s_k)^\lambda$.

To any t.d.o. ring $\mathcal{A}$ one associates a stack of twisted sheaves $\mathfrak{S}_{\mathcal{A}}$ and a twisted line bundle $\mathcal{O}_{\mathcal{A}} \in \mathsf{Mod}(\mathcal{O}_X; \mathfrak{S}_{\mathcal{A}})$, such that

$$\mathcal{A} \simeq \mathcal{O}_{\mathcal{A}} \otimes_{\mathcal{O}} \mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{O}_{\mathcal{A}}^{-1}.$$

The Riemann-Hilbert correspondence associates flat connections of rank 1 in $\mathsf{Mod}(\mathcal{A})$ with local systems of rank 1 in $\mathfrak{S}_{\mathcal{A}}(X)$, by $\mathcal{M} \mapsto \mathcal{H}om_{\mathcal{A}}(\mathcal{M}, \mathcal{O}_{\mathcal{A}})$. Recalling that $\mathfrak{S}_{\mathcal{A}}$ is globally $\mathbb{C}$-equivalent to the stack of sheaves $\mathfrak{Mod}(\mathbb{C}_X)$ if and only if there is a local systems of rank 1 in $\mathfrak{S}_{\mathcal{A}}(X)$, it follows

Proposition 2.1. *Let $\mathcal{A}$ be a t.d.o. ring on X, and $\mathcal{M}$ a flat connection of rank 1 in $\mathsf{Mod}(\mathcal{A})$. Then $\mathfrak{S}_{\mathcal{A}}$ is globally $\mathbb{C}$-equivalent to $\mathfrak{Mod}(\mathbb{C}_X)$.*

We will make use of the exact sequence

$$H^1(X; \mathcal{O}_X^\times) \xrightarrow[\gamma]{} H^1(X; d\mathcal{O}_X) \xrightarrow[\delta]{} H^2(X; \mathbb{C}_X^\times), \tag{2.1}$$

induced by the short exact sequence

$$1 \to \mathbb{C}_X^\times \to \mathcal{O}_X^\times \xrightarrow[d\log]{} d\mathcal{O}_X \to 0.$$

If $\mathcal{L}$ is a line bundle, and $\mathcal{A}$ a t.d.o. ring, one has $\gamma([\mathcal{L}]) = [\mathcal{D}_{\mathcal{L}}]$, $\delta([\mathcal{A}]) = [\mathfrak{S}_{\mathcal{A}}]^{-1}$.

3. Systems with simple characteristics

Let us now recall some definitions and results on microdifferential operators, due to [11, 10]. See also [6, 8] for an exposition.

Let X be a complex manifold, and $\pi\colon T^*X \to X$ its cotangent bundle. Denote by $\mathcal{E}_X$ the ring of microdifferential operators on T^*X, and by $\mathsf{F}_m\mathcal{E}_X$ its subsheaf of microdifferential operators of order at most m.

Let (x) be a system of local coordinates in X, and denote by $(x;\xi)$ the associated system of symplectic coordinates in T^*X. With this choice of coordinates, a microdifferential operator $P \in \mathsf{F}_m\mathcal{E}$ is described by its total symbol $\{p_k(x;\xi)\}_{k\le m}$, where $p_k \in \mathcal{O}_{T^*X}(k)$ is a function homogeneous of degree k. The principal symbol of order m, independent from the choice of coordinates, is given by

$$\begin{aligned} \sigma_m\colon \mathsf{F}_m\mathcal{E} &\to \mathcal{O}_{T^*X}(m) \\ P &\mapsto p_m. \end{aligned}$$

Denoting by $a\colon T^*X \to T^*X$ the antipodal map, the formal adjoint of P is the operator $P^* \in a^{-1}\mathsf{F}_m\mathcal{E}$ whose principal symbol $\{p_k^*(x;\xi)\}_{k\le m}$ is given by

$$p_k^*(x;\xi) = \sum_{k=l-|\alpha|} \frac{(-1)^{|\alpha|}}{\alpha!}(\partial_\xi^\alpha \partial_x^\alpha p_l)(x;-\xi).$$

For $P \in \mathsf{F}_m\mathcal{E}$, one has $P-(-1)^m P^* \in \mathsf{F}_{m-1}\mathcal{E}$. The subprincipal symbol of P in the coordinate system (x) is defined by

$$\begin{aligned} \sigma'_{m-1}(P) &= \frac{1}{2}\sigma_{m-1}(P-(-1)^m P^*) \\ &= p_{m-1} - \frac{1}{2}\sum_i \partial_{x_i}\partial_{\xi_i} p_m. \end{aligned}$$

Identifying X with the zero-section of T^*X, we set $\dot{T}^*X = T^*X \setminus X$. In this paper, by submanifold of T^*X we mean a smooth locally closed submanifold. A submanifold V of T^*X is conic if it is locally $\mathbb{C}^\times$-invariant. The canonical 1-form induces a homogeneous symplectic structure on T^*X. A submanifold V of T^*X is involutive if for any pair f, g, of holomorphic functions vanishing on V, the Poisson bracket $\{f, g\}$ vanishes on V.

Let V be a conic involutive submanifold of $\dot{T}^*X$. The ring $\mathcal{E}_V$ on V is the subring of $\mathcal{E}_X$ generated by

$$\mathcal{J}_V = \{P \in \mathsf{F}_1\mathcal{E}_X \colon \sigma_1(P)|_V = 0\}.$$

The V-filtration is defined by $\mathsf{F}_k^V\mathcal{E}_X = \mathsf{F}_k\mathcal{E}_X \cdot \mathcal{E}_V = \mathcal{E}_V \cdot \mathsf{F}_k\mathcal{E}_X$.

If $\mathcal{M}$ is an $\mathcal{E}_X$-module and $\mathcal{M}_0$ is an $\mathsf{F}_0\mathcal{E}_X$-submodule, we set

$$\mathcal{M}_k = (\mathsf{F}_k\mathcal{E}_X)\mathcal{M}_0.$$

Definition 3.1. One says that a coherent $\mathcal{E}_X$-module $\mathcal{M}$ is globally simple along V if it admits a coherent sub-$\mathsf{F}_0\mathcal{E}_X$-module $\mathcal{M}_0$ such that $\mathcal{M} = \mathcal{E}_X\mathcal{M}_0$, $\mathcal{E}_V\mathcal{M}_0 \subset \mathcal{M}_0$, and $\mathcal{M}_0/\mathcal{M}_{-1}$ is a locally isomorphic to $\mathcal{O}_V(0)$. Such an $\mathcal{M}_0$ is called a V-lattice.

To $P \in \mathcal{J}_V$ one associates the operator $\mathcal{L}_V(P) \in \mathsf{F}_1\mathcal{D}_{\Omega_{V/X}^{1/2}}$ defined as follows. For $\omega \in \Omega_V$ one sets

$$\mathcal{L}_V(P)(\omega^{1/2}/\sqrt{dx}) = \left(L_{H_{\sigma_1(P)}}(\omega^{1/2}) + \sigma_0'(P)\cdot\omega^{1/2}\right)/\sqrt{dx},$$

where dx is the volume form associated with a chosen local coordinate system. The Hamiltonian vector field $H_{\sigma_1(P)}$ is tangent to V since $\sigma_1(P)|_V = 0$. Recall that for $v \in \Theta_V$, one sets $L_v(\omega^{1/2}) = \frac{1}{2}\frac{L_v(\omega)}{\omega}\omega^{1/2}$. The operator $\mathcal{L}_V(\cdot)$ does not depend on the choice of coordinates, and extends as a ring morphism

$$\mathcal{L}_V \colon \mathcal{E}_V \to \mathcal{D}_{\Omega_{V/X}^{1/2}}. \tag{3.1}$$

Denote by $\mathcal{D}_V(0)$ the subring of $\mathcal{D}_V$ consisting of differential operators which commute with the Euler vector field on V, by $\mathcal{D}_V^{bic}$ the subring of $\mathcal{D}_V$ consisting of differential operators which commute with the functions $a \in \mathcal{O}_V$ constant along the bicharacteristic leaves of V, and set $\mathcal{D}_V^{bic}(0) = \mathcal{D}_V^{bic} \cap \mathcal{D}_V(0)$.

The following result is essentially due to [10, 9].

Theorem 3.2. *The ring morphism* (3.1) *induces a ring isomorphism*

$$\mathcal{L}_V \colon \mathcal{E}_V/\mathsf{F}_{-1}\mathcal{E}_V \xrightarrow{\sim} \mathcal{D}^{bic}_{\Omega_{V/X}^{1/2}}(0). \tag{3.2}$$

4. Statement of the main result

Let $\mathcal{A}$ be a t.d.o. ring on X, and V a conic involutive submanifold of $\dot{T}^*X$. All definitions and constructions of §3 extend to the twisted case. In particular, setting

$$\begin{aligned}\mathcal{E}_{\mathcal{A}} &= \pi^{-1}\mathcal{O}_{\mathcal{A}} \otimes_{\pi^{-1}\mathcal{O}} \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{O}} \pi^{-1}\mathcal{O}_{\mathcal{A}}^{-1},\\ \mathcal{E}_{V,\mathcal{A}} &= \pi^{-1}\mathcal{O}_{\mathcal{A}} \otimes_{\pi^{-1}\mathcal{O}} \mathcal{E}_V \otimes_{\pi^{-1}\mathcal{O}} \pi^{-1}\mathcal{O}_{\mathcal{A}}^{-1},\end{aligned}$$

Theorem 3.2 gives an isomorphism

$$\mathcal{L}_V \colon \mathcal{E}_{V,\mathcal{A}}/\mathsf{F}^V_{-1}\mathcal{E}_{\mathcal{A}} \xrightarrow{\sim} \mathcal{D}^{bic}_{\Omega^{1/2}_{V/X}\otimes_{\mathcal{O}}\pi_V^*\mathcal{O}_{\mathcal{A}}}(0), \tag{4.1}$$

where π_V is the restriction to V of $\pi\colon T^*X \to X$.

Let $j_\Sigma\colon \Sigma \rightarrowtail V$ be the embedding of a smooth bicharacteristic leaf of V. For $\mathcal{G}$ a twisted line bundle on V, we consider the restriction functor

$$j_\Sigma^*\colon \mathsf{Mod}(\mathcal{D}^{bic}_{\mathcal{G}}(0)) \to \mathsf{Mod}(\mathcal{D}_{j_\Sigma^*\mathcal{G}}(0)),$$

and the pull-back

$$\pi_\Sigma^\sharp\colon H^2(X;\mathbb{C}_X^\times) \to H^2(\Sigma;\mathbb{C}_\Sigma^\times). \tag{4.2}$$

Recall the maps γ and δ in (2.1).

Theorem 4.1. *Let V be a conic involutive submanifold of $\dot{T}^*X$, and let Σ be a smooth bicharacteristic leaf of V. Let $\mathcal{A}$ be a t.d.o. ring on V, and let $\mathcal{M}$ be a globally simple $\mathcal{E}_{\mathcal{A}}$-module along V. Then*

$$\delta(\pi_\Sigma^\sharp[\mathcal{A}]) = \delta(-\frac{1}{2}\cdot\gamma([\Omega_{\Sigma/X}])) \quad \textit{in } H^2(\Sigma;\mathbb{C}_\Sigma^\times).$$

Sketch of proof. The proof follows the same lines as in [9, §I.5.2]. Let $\mathcal{M}_0$ be a V-lattice in $\mathcal{M}$. By (4.1), $\mathcal{M}_0/\mathcal{M}_{-1}$ is locally isomorphic to $\mathcal{O}_V(0)$ as $\mathcal{D}^{bic}_{\Omega^{1/2}_{V/X}\otimes_{\mathcal{O}}\pi_V^*\mathcal{O}_{\mathcal{A}}}(0)$-modules. Note that $j_\Sigma^*\Omega_V \simeq \Omega_\Sigma$. Then $j_\Sigma^*(\mathcal{M}_0/\mathcal{M}_{-1})$ is a $\mathcal{D}_{\Omega^{1/2}_{\Sigma/X}\otimes_{\mathcal{O}}\pi_\Sigma^*\mathcal{O}_{\mathcal{A}}}(0)$-module which is locally isomorphic to $\mathcal{O}_\Sigma(0)$. The statement follows by Proposition 2.1. □

5. Non existence of twisted wave equations

Let $\mathbb{T} \simeq \mathbb{C}^4$ be a complex vector space, $\mathbb{P}$ the projective space of lines in $\mathbb{T}$, and $\mathbb{G}$ the Grassmannian of 2-dimensional subspaces in $\mathbb{T}$. The Penrose correspondence (see [5]) is associated with the double fibration

$$\mathbb{P} \xleftarrow[f]{} \mathbb{F} \xrightarrow[g]{} \mathbb{G} \tag{5.1}$$

where $\mathbb{F} = \{(y,x) \in \mathbb{P}\times\mathbb{G}; y \subset x\}$ is the incidence relation, and f, g are the natural projections. The double fibration (5.1) induces the maps

$$\dot{T}^*\mathbb{P} \xleftarrow[p]{} \dot{T}^*_{\mathbb{F}}(\mathbb{P}\times\mathbb{G}) \xrightarrow[q]{} \dot{T}^*\mathbb{G},$$

where $T^*_{\mathbb{F}}(\mathbb{P}\times\mathbb{G}) \subset T^*(\mathbb{P}\times\mathbb{G})$ denotes the conormal bundle to $\mathbb{F}$, and p and q are the natural projections. Note that p is smooth surjective, and q is a closed embedding. Set

$$V = q(\dot{T}^*_{\mathbb{F}}(\mathbb{P}\times\mathbb{G})).$$

Then V is a closed conic regular involutive submanifold of $\dot{T}^*\mathbb{G}$, and q identifies the fibers of p with the bicharacteristic leaves of V.

For $m \in \mathbb{Z}$ consider the line bundles $\mathcal{O}_{\mathbb{P}}(m)$, and set

$$\mathcal{M}_{(1+m/2)} = H^0(\mathbb{D}g_*\mathbb{D}f^*(\mathcal{D}_{\mathbb{P}} \otimes_{\mathcal{O}} \mathcal{O}_{\mathbb{P}}(-m))),$$

where $\mathbb{D}g_*$ and $\mathbb{D}f^*$ denote the direct and inverse image in the derived categories of $\mathcal{D}$-module. As we recalled in the Introduction, according to Penrose $\mathbb{G}$ is a conformal compactification of the complexified Minkowski space, and the $\mathcal{D}_{\mathbb{G}}$-module $\mathcal{M}_{(1+m/2)}$ corresponds to the massless field equation of helicity $1+m/2$.

By [3], $\mathcal{M}_{(1+m/2)}$ is globally simple along V.

Theorem 5.1. *Let $\mathcal{A}$ be a t.d.o. ring on $\mathbb{G}$, and $\mathcal{M}$ an $\mathcal{A}$-module globally simple along V. Then $\delta[\mathcal{A}] = 1$ in $H^2(\mathbb{G};\mathbb{C}^\times_{\mathbb{G}})$. In particular,* $\mathsf{Mod}(\mathcal{A})$ *and* $\mathsf{Mod}(\mathcal{D}_{\mathbb{G}})$ *are $\mathbb{C}$-equivalent.*

Proof. By Theorem 4.1 it is enough to show that for a bicharacteristic leaf $\Sigma \subset V$, one has an isomorphism $\pi^\sharp_\Sigma\colon H^1(\mathbb{G}; d\mathcal{O}_{\mathbb{G}}) \xrightarrow{\sim} H^1(\Sigma; d\mathcal{O}_\Sigma)$, and moreover $\delta(-\gamma([\Omega_{\Sigma/\mathbb{G}}])/2) = 1 \in H^2(\Sigma;\mathbb{C}^\times_\Sigma)$.

With the identification

$$T^*\mathbb{G} = \{(x;\xi); x\subset\mathbb{T}, \xi\in \operatorname{Hom}(\mathbb{T}/x, x)\},$$

one has

$$V = \{(x;\xi); \operatorname{rk}(\xi) = 1\},$$

where $\operatorname{rk}(\xi)$ denotes the rank of the linear map ξ. There is an isomorphism

$$\dot{T}^*\mathbb{P} = \{(y,z;\theta); y\subset z\subset\mathbb{T},\ \theta\colon \mathbb{T}/z \xrightarrow{\sim} y\},$$

where y is a line and z is a hyperplane. The projection $q\colon V \to \dot{T}^*\mathbb{P}$ is given by $q(x;\xi) = (\operatorname{im}\xi, x+\ker\xi; \tilde{\xi})$, where $\tilde{\xi}$ satisfies $\tilde{\xi}\circ\ell = \xi$ for $\ell\colon \mathbb{T}/x \twoheadrightarrow \mathbb{T}/x+\ker\xi$ the natural map.

Recall that the bicharacteristic leaves of V are the fibers of q. For $(y,z,\theta)\in\dot{T}^*\mathbb{P}$, the bicharacteristic leaf $q^{-1}(y,z,\theta)$ of V is given by

$$\Sigma = \{(x;\xi); y\subset x\subset z,\ \xi = \theta\circ\ell\},$$

where $\ell\colon \mathbb{T}/x \twoheadrightarrow \mathbb{T}/z$ is the natural map. Thus, Σ is the projective space of lines in z/y. Hence the sequence

$$H^1(\Sigma;\mathcal{O}^\times_\Sigma) \xrightarrow[\gamma]{} H^1(\Sigma; d\mathcal{O}_\Sigma) \xrightarrow[\delta]{} H^2(\Sigma;\mathbb{C}^\times_\Sigma)$$

is isomorphic to the sequence of additive abelian groups

$$\mathbb{Z} \xrightarrow[\gamma]{} \mathbb{C} \xrightarrow[\delta]{} \mathbb{C}/\mathbb{Z},$$

with $1 \in \mathbb{Z}$ corresponding to $[\mathcal{O}_\Sigma(-1)]$.

Denote by $\mathcal{O}_\mathbb{G}(-1)$ the determinant of the tautological bundle on $\mathbb{G}$. Then $H^1(\mathbb{G}; d\mathcal{O}_\mathbb{G}) \simeq \mathbb{C}$ with generator $\mathcal{D}_{\mathcal{O}_\mathbb{G}(-1)}$. Since $\pi_\Sigma^* \mathcal{O}_\mathbb{G}(-1) \simeq \mathcal{O}_\Sigma(-1)$, it follows that

$$\pi_\Sigma^\sharp \colon H^1(\mathbb{G}; d\mathcal{O}_\mathbb{G}) \xrightarrow{\sim} H^1(\Sigma; d\mathcal{O}_\Sigma).$$

There are isomorphisms $\Omega_\mathbb{G} \simeq \mathcal{O}_\mathbb{G}(-4)$, and $\Omega_\Sigma \simeq \mathcal{O}_\Sigma(-2)$, so that $\pi_\Sigma^* \Omega_\mathbb{G} \simeq \pi_\Sigma^* \mathcal{O}_\mathbb{G}(-4) \simeq \mathcal{O}_\Sigma(-4)$, and $\Omega_{\Sigma/\mathbb{G}} \simeq \mathcal{O}_\Sigma(2)$. It follows that

$$[\Omega_{\Sigma/\mathbb{G}}] = 2 \quad \text{in } \mathbb{Z} \simeq H^1(\Sigma; \mathcal{O}_\Sigma^\times),$$

and therefore

$$\delta(-\gamma([\Omega_{\Sigma/\mathbb{G}}])/2) = 0 \quad \text{in } \mathbb{C}/\mathbb{Z} \simeq H^2(\Sigma; \mathbb{C}_\Sigma^\times).$$

By Theorem 4.1, it follows that $\delta(\pi_\Sigma^\sharp[\mathcal{A}]) = 1$. Hence $\delta([\mathcal{A}]) = 1$, so that $\mathfrak{S}_\mathcal{A}$ is globally $\mathbb{C}$-equivalent to $\mathfrak{Mod}(\mathbb{C}_\mathbb{G})$. In particular, $\mathcal{O}_\mathcal{A}$ in an untwisted sheaf. The equivalence

$$\mathsf{Mod}(\mathcal{D}_\mathbb{G}) \xrightarrow{\sim} \mathsf{Mod}(\mathcal{A})$$

is given by $\mathcal{M} \mapsto \mathcal{O}_\mathcal{A} \otimes_\mathcal{O} \mathcal{M}$. □

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